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Deformed Whitham equations for some near-integrable systems

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A perturbation approach for some near-integrable systems with periodic boundary conditions is developed. Deformed Whitham equations including perturbaton terms of a special type are derived in a common form for Ablowitz-Kaup-Newell-Segur models. This approach is used for analysis of the generation of a dense packet of solitons in a model of a two-level laser with pumping of the upper level and relaxation.
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I. INTRODUCTION

The description of soliton generation in nonlinear media is an interesting and important problem in theoretical physics [1–3]. The generation of ultrashort pulses in the amplifying systems has also attracted researchers' attention. The dynamics of laser pulses isolated from each other has been studied in many publications. Frequently, however, the initial stage of generation involves a high density of solitons or another type of nonlinear pulse. This situation may arise, for instance, in a powerful laser with small losses [4]. The behavior and characteristics of solitons depend on mutual interaction of solitons in a dense packet. Knowledge of such characteristics is important in the application of generated pulses in nonlinear optics and also for improving the effectiveness of laser systems. To gain information on output pulse characteristics, one has to establish an onset of pulses and regimes of generation.

The most detailed information about evolution of fields in nonlinear models may be obtained by using the inverse scattering transform (IST) [5]. The dynamics of solitons in isolation in the attenuators are now well understood mainly owing to application of the IST to the solution of nonlinear models. The characteristics of generated pulses was studied in the framework of the IST by Manakov and co-workers for a long two-laser amplifier [6]. As considered in the Ref. [6] models, asymptotics of laser fields are described using a similar solution. Analogous asymptotics may be realized in a mathematically related model of stimulated Raman scattering [7].

More often than not, another scheme of amplification is used in lasers; for details, see [1,4]. An idealized powerful laser with small losses may be simulated using the Maxwell-Bloch equations for a two-level system with pumping of an upper level. This and related models have been studied in many publications. However, a detailed description of the dynamics of a dense packet of generated solitonlike ultrashort pulses is, to our knowledge, absent in the literature.

To treat a nonlinear stage of evolution of the dense packets of pulses, one must operate with a large number of degrees of freedom. Such treatment is possible, as a rule, only in completely integrable models, and even for the completely integrable systems it leads to tremendous analytical problems. On the other hand, some experimental results of the generation of dense packets of pulses may be described using modulated periodic waves. Furthermore, experimental and numerical studies of soliton generation in lasers with small losses show that an initial stage of generation may be simulated using one- (two-) phase waves with modulated shapes [4]. It worth mentioning that similar situations arise in studying the evolution of steplike fields in attenuators [5]. This observation motivates one to use a heuristic Whitham approach to study the behavior of dense packets of pulses. This approach consists of two steps. The first step is a derivation of an exact one- (two-) phase solution to the original equations with periodic boundary conditions. Then it is assumed that spectral data associated with periodic wave can depend on space and time variables. This dependence is slower than that of a single oscillation consisting of a single packet. Averaging over fast nonlinear oscillations yields evolution equations for parameters of the periodic wave. These equations are the hydrodynamic Whitham equations [8,5], which can be effectively obtained using the IST [12].

Perturbation theory can be developed for systems with periodic boundary conditions as well. It leads, usually, to a cumbersome theory. In the present paper we develop effective perturbation theory for slowly modulating dense packets of pulses. A peculiarity of this theory is the possibility of incorporating the perturbations by means of prolongation of derivatives of a particular type. The ultimate goal of this paper is the development of a perturbation approach and its application to the description of slowly modulating periodic solutions to the near-integrable system. This approach is based on the IST technique, which allows one to receive modified Whitham equations for slowly changing parameters of periodic waves with terms describing perturbations directly in diagonal form.

In Sec. II an approach is developed for a common near-integrable Ablowitz-Kaup-Newell-Segur (AKNS) system. Results obtained by Burtsev, Mikhailov, and Zakharov [9] for exactly integrable models with changing spectral parameters are used. Deformed Whitham equations are derived in Sec. III by means of the extended method of Flaschka, Forest, and McLaughlin [12]. Conventional Whitham equations are modified to include perturbations terms.

The approach developed here is used for the study of soliton train generation in a model two-level laser. This

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model may be used to describe an initial stage of ultrashort pulse generation in gas, dye, and solid state lasers [4]. It is known that some kind of pumping and relaxation can be included in the IST without the IST losing integrability. Generalization of models may be done if it is assumed that the spectral parameter depends on variables [9,10]. A corresponding modification of the IST can be effectively used to study isolated pulse dynamics. It is known, however, that both relaxation effects and pumping can be included in the above laser model without loss of applicability of the IST only if artificial unphysical conditions are fulfilled; for details, see [10]. These conditions are avoided and additional perturbation terms can be treated in the approach presented here. Section IV is devoted to studying the generation of dense packets of pulses. Exactly integrable two-level laser models with perturbations describing relaxation and pumping effects are investigated. Two different solutions to modified Whitham equations are derived and investigated. These solutions demonstrate transformation of an initial constant field having small amplitude in a sequence of solitons. The influence of pumping and relaxation on the characteristics of generated solitons is found and compared with experimental data.

II. THEORY OF DEFORMED NEAR-INTEGRABLE MODELS

In this section, we extend the IST with variable spectral parameters to near-integrable equations. Nonisospectral evolution equations arise as a compatibility condition of the following linear systems:

$$\Phi_\xi = U\Phi,\qquad \Phi_\eta = V\Phi. \tag{2.1}$$

Here U, V, and Φ are the matrix $N\times N$ complex-valued functions of ξ, η, and the spectral parameter λ. In general, U and V depend on λ rationally,

$$U(\lambda,\xi,\eta)=u_0+\sum_{n=1}^{N_1}\frac{u_n(\xi,\eta)}{\lambda-\lambda_n},\qquad V(\lambda,\xi,\eta)=v_0+\sum_{n=1}^{N_2}\frac{v_n(\xi,\eta)}{\lambda-\mu_n}, \tag{2.2}$$

where simple poles λ_n and μ_m do not coincide. In the conventional IST, λ, λ_n, μ_n are assumed to be constants. Let poles λ_n,μ_n be functions of ξ,η and depend on hidden parameter ζ. The compatibility condition

$$U_\eta - V_\xi + [U,V]=0 \tag{2.3}$$

must be fulfilled precisely over ζ and the related system of nonlinear equations for the matrices u_n, v_n possess exactly gauge indeterminacy. In exact theories this fact imposes restrictions on λ, which can be found in explicit form. Consider the following generalization of compatibility condition

$$D_\eta U - D_\xi V + [U,V]=0, \tag{2.4}$$

where $D_\eta=(\partial/\partial\eta)+F(\lambda,\eta,\xi)(\partial/\partial_\lambda)$, $D_\xi=(\partial/\partial\xi)+G(\lambda,\eta,\xi)(\partial/\partial_\lambda)$. The condition of compatibility (2.4) requires that the following relation be held for the exactly integrable models

$$F_\eta + GF_\lambda = G_\xi + FG_\lambda. \tag{2.5}$$

Relation (2.5) was derived by Burtsev, Mikhailov, and Zakharov [9] for the exactly integrable models. Restrictions corresponding to the η,ξ dependence on λ is found in Ref. [9]. We aim to construct the perturbation theory using an extension of the above results to near-integrable systems. We consider perturbations that satisfy condition (2.4) in the first approximation. Let us show that this condition can be approximately fulfilled for some class of near integrable systems. Consider the following form of functions F and G:

$$F=\varepsilon f(\varepsilon\eta,\varepsilon\xi,\lambda),\quad G=\varepsilon g(\varepsilon\eta,\varepsilon\xi,\lambda), \tag{2.6}$$

where ε is a small parameter. This means that both functions F and G are small and slow functions of variables η and ξ. Dependence of these functions on spectral parameter λ may be arbitrary. Additionally, λ may be a slow function of η and ξ. Direct substitution of Eq. (2.6) into (2.5) shows that condition (2.5) is fulfilled for arbitrary f and g if one neglects the terms of order ε^2. Thus, under this approximation one can include in evolution equations terms describing perturbations having order ε. Compatibility condition (2.4) is satisfied up to terms having order ε^2.

Instead of the above prolongation of partial derivatives, one can use a variable-dependent spectral parameter λ, which has to obey to the following pair of equations:

$$\partial_\eta\lambda = -F(\lambda),\qquad \partial_\xi\lambda=-G(\lambda). \tag{2.7}$$

For slowly changing $\lambda=\lambda(\varepsilon\eta,\varepsilon\xi)$ relations (2.5) are fulfilled for any λ_m, c_m, b_m if one neglects terms of order ε^2.

Let the functions F and G have the following forms:

$$F=-\varepsilon\sum_{m=1}^{N_1}\frac{c_m}{\lambda-\lambda_m},\quad G=-\varepsilon\sum_{m=1}^{N_2}\frac{b_m}{\lambda-\lambda_m}. \tag{2.8}$$

Then the general perturbed equations have the following form:

$$\partial_\eta u_0-\partial_\xi v_0+[u_0,v_0]=0.$$

$$\frac{\partial u_n}{\partial\eta}+\left[u_n,\sum_{k=1}^{N2}\frac{v_k}{\lambda_n-\mu_k}\right]=\varepsilon\sum_{m=1}^{N_2}\frac{b_m u_n+c_n v_m}{(\lambda_n-\mu_m)^2}, \tag{2.9}$$

$$\frac{\partial v_n}{\partial\xi}+\left[v_n,\sum_{k=1}^{N_1}\frac{u_k}{\mu_n-\lambda_k}\right]=\varepsilon\sum_{m=1}^{N_1}\frac{c_m v_n+b_n u_m}{(\lambda_m-\mu_n)^2}.$$

It is known that for the exact model, conditions (2.8) impose rigorous restrictions on the trajectory of λ [9]. The perturbation approach based on the above formulas allows one to avoid such restrictions. At the same time, the form of perturbations terms used above allows one to take advantage of the IST.

Applicability of the perturbation theory is restricted to special perturbation forms. These perturbations have to be slow functions of variables η and ξ. Perturbation terms may include the effects of relaxation or loss for fields and material equations, different forms of pumping, variation of media parameters, and so on. These perturbations result in a slow change in parameter solutions. For instance, for the nonlinear Schrödinger equations, such perturbation terms may describe slow time and space modulation of medium density. Using the example of the two-level laser model, it will be shown in Sec. IV that these perturbations may be of actual physical interest.

III. DEFORMED WHITHAM EQUATIONS

In this section, the above perturbation approach will be used to construct the perturbation theory for the periodic solutions to the evolution systems. Generalization or ''deformation'' of the Whitham equations can be derived by straightforward application of the IST with prolonged derivatives. (The term ''deformation'' is borrowed from the paper of Burtsev, Mikhailov, and Zakharov [9]). The Whitham equations for AKNS systems can be directly derived in the framework of the IST in diagonal form. It will be shown below that in the framework of the approach presented here, the deformed Whitham equations have a diagonal form as well.

Smoothed shock waves or any modulated wave train may be described in a quasiclassical (or hydrodynamic) approximation. In this approximation, it is assumed that the length and duration of a train or region of oscillations is much more than that of each soliton or nonlinear pulsation filling the region of oscillations. It is suggested that the characteristic parameters of the periodic solution are slow functions of variables ξ,η. These parameters are the roots of polynomial P (see below). Their scales of change are much greater than that of the single pulse. These slowly changing parameters obey equations that may be found by averaging some integrals over the period of fast pulsations T. In this way, analysis of a complex system with many degrees of freedom is reduced to the solution of a few evolution equations.

We derive deformed Whitham equations using an extension of the approach of Flaschka, Forest, and McLaughlin [12] to the near-integrable models under consideration. As mentioned above, an exact method of solving the evolution equations under periodic boundary conditions is very effective for deriving the modulation Whitham equations. Let the AKNS system have the following Lax representation:

$$D_\xi\Phi=\hat{L}\Phi=\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & -L_{11}\end{pmatrix}\Phi, \tag{3.1}$$

$$D\eta\Phi=\hat{A}\Phi=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & -A_{11}\end{pmatrix}\Phi. \tag{3.2}$$

where $\hat{A}$ and $\hat{L}$ are the functions of fields and spectral parameter λ. Φ is a matrix-valued function. Following the approach developed in Refs. [13], we introduce the quadratic eigenfunctions

$$f=(i/2)(\phi_1\psi_2+\phi_2\psi_1),\quad g=\phi_1\psi_1,\quad h=\phi_2\psi_2, \tag{3.3}$$

where $\phi_{1,2}$ and $\psi_{1,2}$ are the different solutions of system (3.1), (3.2). These functions satisfy the system

$$\begin{gathered} D_\eta f=i(A_{12}h-A_{21}g),\quad D_\xi f=i(L_{12}h-L_{21}g),\\ D_\eta g=2iA_{12}f+2A_{11}g,\quad D_\xi g=2iL_{12}f+2L_{11}g, \\ D_\eta h=-2iA_{21}f-2A_{11}h,\quad D_\xi h=-2iL_{21}f-2L_{11}h. \end{gathered}\tag{3.4}$$

It can be easily checked from system (3.4) that value $P(\lambda)=f^2-gh$ satisfies the conditions $D_\xi P(\lambda)=0$, $D_\eta P(\lambda)=0$.

The shape of the periodic solution is determined by the dependence of P on spectral parameter λ. For instance, the N-phase solution is determined by the following polynomial dependence:

$$f^2-gh=P(\lambda)=\prod_{k=1}^{2N+2}(\lambda-\lambda_k)=\sum_{j=0}^{2N+2}P_j\lambda^j. \tag{3.5}$$

Here, λ_k are the roots of polynomial $P(\lambda)$ fixed by the initial conditions. The meaning of the coefficients P_k depends on a particular form of the Lax pair. It can be shown that quadratic functions, satisfying the system (3.1), (3.2), have for the N-phase case the following forms:

$$f=\sum_{k=o}^{N+1}f_k\lambda^{2k},\quad g=l(\lambda)\sum_{k=1}^{N+1}g_k\lambda^k,\quad h=p(\lambda)\sum_{k=1}^{N+1}h_k\lambda^k, \tag{3.6}$$

where $l(\lambda),p(\lambda)$ are some functions of spectral parameter λ determined by the Lax representation. $l(\lambda)$ [$p(\lambda)$] is a common multiplier of L_{12} and A_{12} (L_{21} and A_{21}); $l(\lambda)$ and $p(\lambda)$ are usually related by symmetry conditions. Following the Marchenko approach [13,15] assume that

$$g=l(\lambda)\prod_{k=1}^{N+1}[\lambda-\mu_k(\xi,\eta)], \tag{3.7}$$

where μ_k are the auxiliary spectrum points. Substituting expression (3.7) in equations (3.4) and setting $\lambda=\mu_k$ we get a set of equations for μ_k,

$$D_\xi\mu_k=\frac{2iL_{12}(\mu_k)\sqrt{P(\mu_k)}}{\prod_{j\neq k}{}_{j\neq k}(\mu_k-\mu_j)l(\mu_k)}, \tag{3.8}$$

$$D_\eta\mu_k=\frac{2iA_{12}(\mu_k)\sqrt{P(\mu_k)}}{\prod_{j\neq k}(\mu_k-\mu_j)l(\mu_k)}. \tag{3.9}$$

The approach consisting of derivation of a set of conservation laws in the AKNS scheme is based on the following identity:

$$D_\eta\left(\frac{L_{12}}{g}\right)=D_\xi\left(\frac{A_{12}}{g}\right), \tag{3.10}$$

which is a direct extension of the corresponding identity derived in Refs. [12,15] to a case of prolonged derivatives. Following Forest and McLaughlin [15], we introduce a new normalization $f^2-gh=1$. Then the above identity (3.10) can be rewritten in the following form:

$$D_\eta \frac{L_{12}(\lambda)\sqrt{P(\lambda)}}{\prod_{k=1}^{N+1}(\lambda-\mu_k)l(\lambda)}=D_\xi \frac{A_{12}(\lambda)\sqrt{P(\lambda)}}{\prod_{k=1}^{N+1}(\lambda-\mu_k)l(\lambda)}. \quad (3.11)$$

Averaging the above equations over the period of oscillations yields the Whitham equations for slowly changing spectral parameters λ_k. Using relations (3.8),(3.9), averaging over fast variables ξ,η can be replaced by integration on μ_k [12]. The angle brackets denote this averaging. The vanishing of singularities in the limits $\lambda\to\lambda_k$, where λ_k are the branch points of $\sqrt{P(\lambda)}$, directly yields the Whitham equations in a diagonal form:

$$\frac{\partial\lambda_k}{\partial\eta}+V_k\frac{\partial\lambda_k}{\partial\xi}+F(\lambda_k)+V_kG(\lambda_k)=0, \quad (3.12)$$

where

$$V_k=\left\langle \frac{2iL_{12}(\mu_k)}{\prod_{j\neq k}(\mu_k-\mu_j)l(\mu_k)}\right\rangle \left\langle \frac{2iA_{12}(\mu_k)}{\prod_{j\neq k}(\mu_k-\mu_j)l(\mu_k)}\right\rangle^{-1}. \quad (3.13)$$

Averaging may be done using equations (3.8),(3.9). Integration on phase $W_j=k_j\xi+\omega_k\eta+w_0$, where w_0, k_j, and ω_k are constants, can be changed by integration on auxiliary variables μ_j. Equations (3.8),(3.9) have, on the left-hand sides, derivatives of $\lambda=\mu_k$. Here we aim to apply these equations to derivation of the Whitham equation in the first approximation in degrees of ε, i.e., the terms of order $O(\varepsilon^2)$ must be neglected. This approximation is self-consistent if it is assumed that λ_k are slow functions of variables $\lambda_k=\lambda_k(\varepsilon\eta,\varepsilon\xi)$. The latter assumption means that the derivatives of λ_j of variables ξ,η have the same order as the perturbation terms associated with terms $F(\lambda),G(\lambda)$. Under this assumption, the terms in Eqs. (3.8),(3.9) containing small functions $F(\lambda),G(\lambda)$ yield contributions of order ε to velocities V_k and they have to be neglected in the first approximation in the final deformed Whitham equations.

IV. TWO-LEVEL LASER WITH CONTINUOUS PUMPING

Consider a physical application of the above general theory. It is known that in long-laser amplifiers small initial seed fields evolve into a set of pulses described by a similarity solution [6]. Generally, the structure of the solution is determined by the initial and boundary conditions, perturbations, the form of pumping, and so on. It is important in laser constructions to know the change of pulse velocity with time, the dependence of the pulse shape on the parameters of the seed field, and so on. For lasers with external continuous pumping and small losses, a detailed investigation of the nature of the generated pulses with respect to evolution model may be performed using the above approach. In this section we will investigate two regimes of generation of the soliton trains. The first regime is a nonstationary transformation of long steplike pulses to a consequence of slowly amplifying asymptotic solitons. The second regime consists of transforming a plane wave having initially small amplitude into a set of solitons. The parameters of these solitons tend to be stationary values corresponding to a solitonic attractor. We shall show that the onset of generated pulses may differ from that of long-laser amplifiers considered in [6]. Comparison of qualitative theoretical results with known experimental results seems to be satisfactory.

Let us consider a two-level model with pumping of an upper level. Maxwell-Bloch equations describing resonant interaction of light pulses with atomic transition are as follows:

$$\partial_\tau Q+\gamma_2 Q+2i\nu Q=-\frac{id_{12}}{\hbar}EN, \quad (4.1)$$

$$\partial_\tau N_3+\gamma_1(N_3-N_0)=\frac{id_{12}}{2\hbar}(Q^*E-QE^*)+C_0, \quad (4.2)$$

$$\partial_z E=iN_0\frac{2\pi\omega_0}{c_l}\langle Q\rangle_\Gamma, \quad (4.3)$$

where d_{12} is the dipole momentum of transition, N_3 is the difference between level populations, Q is an off-diagonal part of the density matrix, N_0 is the density of resonant atoms, $\gamma_{1,2}$ are the relaxation constants, τ is the retarded time, ω_0 is the frequency of transition, c_l is the light velocity, and ν is the detuning frequency. Angle brackets denote averaging of polarization for inhomogeneously broadened atoms: $\langle Q\rangle_\Gamma=\int_{-\infty}^{\infty}Q(\nu)\Gamma(\nu)d\nu$ over the detuning distribution $\Gamma(\nu)$. We consider the case in which the time scale of amplitudes of fields is less then $\gamma_{1,2}^{-1}$. For a large family of lasers, pumping may be modeled by including a real function $C_0(x)$. Here $t=\tau\sqrt{\Omega}$, $x=z\sqrt{\Omega}/c_l$, $\Omega=2\pi N_0\omega_0 d_{12}/\hbar$, and $c'(x)=C_0(x)/\sqrt{\Omega}$.

Let us apply the procedure described in the preceding sections to a case of one-phase solution to Maxwell-Bloch equations. The Lax representation of the model has the well known form (3.1),(3.2), where

$$L_{11}=-i\lambda,\quad L_{12}=\frac{E}{2},\quad L_{21}=\frac{E^*}{2},$$

$$A_{11}=\frac{i}{4}\left\langle\frac{N_3}{\lambda-\nu}\right\rangle_\Gamma,\quad A_{12}=-\frac{i}{4}\left\langle\frac{Q}{\lambda-\nu}\right\rangle_\Gamma, \quad (4.4)$$

$$A_{21}=-\frac{i}{4}\left\langle\frac{Q^*}{\lambda-\nu}\right\rangle_\Gamma.$$

Consider the case of $F(\lambda)=\varepsilon[c(\varepsilon x)/\lambda]$, $G(\lambda)=\varepsilon\gamma\lambda$. Here $c'=\varepsilon c,\gamma_1=\gamma_2=\varepsilon\gamma$. Slow dependence of function c on variable x may be arbitrary. Function $F(\lambda)$ describes pumping and function $G(\lambda)$ is related to relaxation effects. We restrict ourselves to the case of equal relaxation constants $\gamma_1=\gamma_2$. This restriction can be fulfilled in a gas of metal atoms. It is known that each of these terms can be implemented in the IST with variables depending on spectral parameter [10]. Relaxation and pumping terms can be unified only if very artificial conditions are imposed on functions γ and c. These conditions are not fulfilled in real laser experiments. Using the approximate approach developed here, we can avoid these restrictions. We are able to investigate a region of parameters that cannot be reached by reduction of the exact nonisospectral IST. Conditions of small pump and relaxation terms can achieved in a modern experimental setup. As will be shown, there are regimes that influence both pumping and relaxation and lead to qualitatively new behavior of generated pulses. Note that the above perturbation approach can be used, in general, for any slow dependence of functions $c(\varepsilon x,\varepsilon t),\gamma(\varepsilon x,\varepsilon t)$.

We use for modeling of the initial stage of laser pulse generation the simplest one-phase solution to the Maxwell-Bloch equation. The exact common N-phase solution to the Maxwell-Bloch equations can be found in terms of θ-functions following [13] for $c=\gamma=0$. The common solution has the following form:

$$E(x,t)=E(0,0)\mathscr{l}^{ik_0x-i\vartheta_0t}\frac{\theta(W^-,\varsigma)}{\theta(W^+,\varsigma)},\qquad(4.5)$$

where $W^{\pm}=(1/2\pi)(k_jx+\Omega_jt+\delta_j^{\pm}),k_j,\Omega_j,\delta_j^{\pm}$ are constants, and ς is a matrix of periods; see, for instance, [14]. The structure of the solution is the same as for the nonlinear Schrödinger equations [13,14] up to its dependence Ω_j on the roots of polynomial P. A one-phase solution to the nonlinear Schrödinger equation can be used for our purposes after changing of the coefficient Ω_1. We shall use this exact one-phase solution as a zero approximation. Perturbation terms yield modulation of a train of the nonlinear pulsations. This modulation is assumed to be of the same order as the dispersion effects described in the hydrodynamic approximation by nonperturbed Whitham equations.

For two pairs of complex conjugated roots of the polynomial P, $\lambda_{1,3}=\alpha\pm i\beta,\lambda_{2,4}=\alpha_0\pm i\beta_0$, we have from Eq. (4.5) the following expression for the intensity of the electric field:

$$|E(x,t)|^2=|E(0,0)|^2[(\beta+\beta_0)^2-4\beta\beta_0 sn^2\{[(\beta+\beta_0)^2+(\alpha-\alpha_0)^2]^{1/2}W,\kappa\}],\qquad(4.6)$$

where κ is a modulus of the Jacobi function and $\kappa^2=(4\beta\beta_0)/[(\beta+\beta_0)^2+(\alpha-\alpha_0)^2]$. Phase W is found from Eqs. (3.8),(3.9). For the one-phase solution it can be done easily:

$$W=2(t+xV_0^{-1}+t_0),\quad V_0=-\int_{-\infty}^{\infty}\frac{\Gamma(\nu)}{4\prod_{i=1}^{4}(\lambda_i-\nu)}d\nu.\qquad(4.7)$$

For the one-phase solution we have to set $N=2,l(\lambda)\equiv1$ in formulas (3.4),(3.6). Then we have

$$D_x\left(\frac{E}{2g}\right)=D_t\left(\frac{A_{12}}{g}\right),\qquad(4.8)$$

where $D_x=\partial_x+F\partial_\lambda,D_t=\partial_t+G\partial_\lambda$. After a new normalization for functions f,g,h such that $f^2-hg=1$, the above equation transforms into the following:

$$D_x\left[\frac{P(\lambda)^{1/2}}{\lambda-\mu}\right]=D_t\left[\frac{P(\lambda)^{1/2}}{V_0}\left(\frac{1}{\lambda-\mu}-\left\langle\frac{1}{\lambda-\nu}\right\rangle\right)\right].\qquad(4.9)$$

The angle brackets in the above relation denote averaging on fast oscillations over the period T. As noted above, averaging can be done by changing of integration on phase W to integration on the auxiliary variable μ. To accomplish this, rewrite formulas (3.8),(3.9) in a common equation

$$\frac{\partial\mu}{\partial W}=-i\sqrt{P(\mu)}+\frac{\varepsilon}{2}\left(\frac{c}{\mu}+\gamma\mu\right).\qquad(4.10)$$

The period of oscillations T is determined by the following integral:

$$T=\int dW=\int\frac{d\mu}{\frac{\varepsilon c}{2\mu}+\frac{\varepsilon\gamma\mu}{2}+\sqrt{P(-\mu)}},\qquad(4.11)$$

Relation (4.11) can be easily obtained from (4.10). If we neglect the perturbation terms $(c=\gamma=0)$ we obtain $T=2K(k)[(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)]^{-1/2}$, where $K(k)$ is a complete elliptic integral of the first kind with modulus k: $k^2=[(\lambda_1-\lambda_2)(\lambda_3-\lambda_4)]/[(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)]$, λ_k are the roots of polynomial P such that $\lambda_1>\lambda_2>\lambda_3>\lambda_4$ [16]. Integration is performed along the curve whose circle cuts between λ_1 and λ_2 or λ_3 and λ_4. Averaging over the period of fast oscillations T is performed by using the following relations:

$$\left\langle\frac{1}{\lambda-\mu}\right\rangle=\frac{1}{T}\int\frac{1}{\lambda-\mu}d\theta=\frac{1}{T}\int\frac{1}{\lambda-\mu}\frac{d\mu}{\frac{\varepsilon c}{2\mu}+\frac{\varepsilon\gamma\mu}{2}+\sqrt{P(-\mu)}}.\qquad(4.12)$$

In perturbation theory, we neglect terms of order ε^2; therefore, terms in Eq. (4.12) of order ε can be avoided because they give a correction of order ε^2 in the final form of the deformed Whitham equations. Thus the terms $(\varepsilon c/\mu),\varepsilon\gamma\mu$ in the above relation have to be avoided.

Setting successively $\lambda=\lambda_n,n=1-4$, we obtain from Eqs. (4.11) and (4.12)

$$\lim_{\lambda\to\lambda_n}\left\langle\frac{1}{\lambda-\mu}\right\rangle=-2\partial_{\lambda_n}(\ln T).\qquad(4.13)$$

The limits $\lambda\to\lambda_n$ yield the singularities in the differentials

$$D_t\sqrt{P(\lambda)},\quad D_x\sqrt{P(\lambda)}.$$

The vanishing of the corresponding coefficients in (4.6) are fulfilled if the spectral parameters obey the following generalization of the Whitham equations:

$$\partial_x\lambda_n-\frac{1}{V_n}\partial_t\lambda_n=\varepsilon\left(\frac{\gamma}{V_n}\lambda_n+\frac{c}{\lambda_n}\right),\qquad(4.14)$$

where

$$\frac{1}{V_n}=\frac{1}{V_0}\left[1-\left(\lambda_n\left\langle\frac{1}{\lambda_n-\mu}\right\rangle\right)^{-1}\right],\qquad(4.15)$$

$$\left\langle \frac{1}{\lambda_1-\mu} \right\rangle = \frac{(\lambda_2-\lambda_3)E(k)+(\lambda_1-\lambda_2)K(k)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)K(k)},$$

$$\left\langle \frac{1}{\lambda_2-\mu} \right\rangle = \frac{(\lambda_1-\lambda_2)K(k)-(\lambda_1-\lambda_4)E(k)}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_4)K(k)},$$

$$\left\langle \frac{1}{\lambda_3-\mu} \right\rangle = \frac{(\lambda_2-\lambda_4)E(k)-(\lambda_2-\lambda_3)K(k)}{(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)K(k)},$$

$$\left\langle \frac{1}{\lambda_4-\mu} \right\rangle = \frac{(\lambda_1-\lambda_3)E(k)-(\lambda_1-\lambda_4)K(k)}{(\lambda_1-\lambda_4)(\lambda_3-\lambda_4)K(k)},$$

where V_0 is written in Eq. (4.7). For $\Gamma(\nu)=\delta(\nu)$ we have $V_0=1/4\sqrt{P_0}$, where $\sqrt{P_0}=+(\lambda_1\lambda_2\lambda_3\lambda_4)^{1/2}$. The sign + has been chosen to make the phase velocity of the pulses less then that of light. $E(k)$ is a complete elliptic integral of the second kind with the same modulus as the above k. From the above derivation, it is obvious that the modified Whitham equations for different AKNS systems have the same view up to its dependence on a phase velocity V_0.

Evolution equations (4.14) in partial derivatives posseses different kinds of solutions. Let us find a nonstationary solution to the Whitham equations (4.14). Suppose that γ depends only on t and c depends on x. Now perform the following substitution:

$$\lambda_k = e^{-\int \gamma' dt}\sqrt{\zeta_k(x,t)+2\int c'(x)dx}, \qquad (4.16)$$

where $\gamma'=\varepsilon\gamma$. The terms on the right-hand side of Eq. (4.14) containing γ and c disappear and Eqs. (4.14) reduce to nondeformed Whitham equations for hidden parameter $\zeta_k(x,t)$. We shall use the substitution (4.16) to analyze splitting of the initial seed field having the form of a steplike pulse into a set of amplifying pulses. The Whitham equations are divided into the two pairs of complex-conjugate equations. Two roots are initially fixed by linking with plane-wave asymptotics at $-\infty$ or $+\infty$; two others roots can be linked with parameters of the highest soliton of the packet of pulses arising near the leading or back edge, respectively. We assume that there are initial conditions leading to the generation of solitons at one of the edges of the steplike pulse without amplification. The most interesting regime for application to laser optics is that a leading edge is associated with the highest soliton. For the Maxwell-Bloch model, such a situation can be realized for some region of parameters of the initial steplike pulse if the system was initially partly inverted. Another method consists in using a seed field having the form of a periodic wave of a special type. Detailed investigation of this problem is not the aim of this paper. We only mention that such regime, the generation of higher solitons, has been observed in many studies. Under some conditions, the formation of solitons near the back edge may be of potential experimental interest as well. Consider a situation in which nonlinear pulses are generated near the back edge of the steplike pulse and tend asymptotically to a set of solitons. It is natural to suppose that the stage intermediate between the plane-wave regime and the solitonic one can be described by a one-phase solution with slowly changing parameters. Such arguments may justify the application of the heuristic Whitham approach to describing the development of modulation instability in attenuators. This approach had been very effectively used for investigation of the transformation of steplike pulses into a set of solitons for the Korteveg–de Vries equation [5]. For the model under consideration, we have made numerous studies and found that under the above-mentioned conditions this approach yields satisfactory results.

The solution to the Whitham equations (4.14) can be found only in an implicit form. Let two roots ζ_1 and ζ_3 be fixed. Consider the most interesting case, in which $\zeta_1=\alpha_0+i\beta_0, \zeta_3=\alpha_0-i\beta_0, \alpha_0,\beta_0\neq 0$. The dynamics of the two "moving" roots ζ_2 and ζ_4 will obey the hydrodynamics approximation to the Whitham equations (4.14). The solution to Eqs. (4.14) fixes the trajectories of roots in the complex plane. Analysis of these trajectories allows one to describe the transformation of weak quasilinear modulation of the plane wave to a set of quasi-isolated solitons. Let $\zeta_2=\alpha+i\beta, \zeta_4=\alpha-i\beta$. The second Whitham Eq. (4.14) (for ζ_2) is the following:

$$\frac{t}{x}=\frac{1}{\sqrt{P_0}}\left\{1-\frac{1}{\alpha+i\beta}\,\frac{2i\beta[\alpha_0-\alpha+i(\beta_0-\beta)]K(\kappa)}{[\alpha_0-\alpha+i(\beta_0-\beta)]K(\kappa)-[\alpha_0-\alpha+i(\beta_0+\beta)]E(\kappa)}\right\}. \qquad (4.17)$$

For the real and imaginary parts of Eq. (4.17) we have

$$R(\kappa)=\frac{E(\kappa)}{K(\kappa)}=\frac{\alpha(\alpha_0^2+\alpha^2+\beta_0^2+\beta^2)-2\alpha(\alpha_0\alpha+\beta_0\beta)}{\alpha(\alpha_0^2+\alpha^2+\beta_0^2+\beta^2)-2\alpha_0(\alpha^2+\beta^2)}, \qquad (4.18)$$

$$\left(\frac{t}{x}\sqrt{P_0}-1\right)(\alpha^2+\beta^2)(\alpha_0-\alpha)^2[1-R(\kappa)]+[\beta_0-\beta+(\beta_0+\beta)R(\kappa)]^2=4\beta(\alpha_0-\alpha)(\alpha_0\beta-\alpha\beta_0)[1-R(\kappa)]$$
$$+[\beta_0-\beta+(\beta_0+\beta)R(\kappa)](\beta\beta_0-\beta^2+\alpha\alpha_0-\alpha^2). \qquad (4.19)$$

Equations (4.18),(4.19) can be solved for α and β as functions of $R(\kappa)$ and the modulus κ,

$$\kappa^2 = \frac{4\beta\beta_0}{(\alpha_0-\alpha)^2+(\beta_0+\beta)^2}. \tag{4.20}$$

Calculations yield

$$\alpha = \frac{\alpha_0}{\alpha_0^2\kappa+\beta_0^2M^2}(\alpha_0^2\kappa^2+(2-\kappa)\beta_0^2M+\beta_0\{2\alpha_0^2\kappa^2(2-\kappa)M+M^2[4\beta_0^2(1-\kappa)-\alpha_0^2\kappa^2]-\alpha_0^2\kappa^2\}^{1/2}), \tag{4.21}$$

$$\beta = \frac{\beta_0\alpha M}{\alpha_0\kappa}, \tag{4.22}$$

where $M(\kappa)=[(2-\kappa)-2(1-\kappa)R(\kappa)]$.

Equations (4.21), (4.22) show that trajectories of roots are strictly determined. This is a consequence of the fact that the system remains Hamiltonian in spite of the fact that spectral data change with changes of variables. Note that spectral data dependence described by the Whitham equations destroys the exact integrability.

Trajectories of the roots $\zeta_2\to\zeta_1$, $\zeta_4\to\zeta_3$ as functions of x/t consist of two symmetric monotonic curves with respect to real axis. The curve lying in an upper half plane starts from the real axis and monotonically tends to a ''top'' value ζ_1 of the imaginary part of the spectrum. The case of coalescing roots $\zeta_2=\zeta_4$ corresponds to the plane-wave limit, from which we started. The solitonic limit is achieved, then $\kappa\to 1$. It may be shown using Eqs. (4.21),(4.22) that root $\zeta_2\to\zeta_1$ and $\zeta_4\to\zeta_3$ with increasing t/x. Near the highest soliton containing the back edge of the packet, neglecting the terms of order of $(1-\kappa)$, we obtain

$$\zeta_2 = \zeta_1\left[1+\frac{2\,\mathrm{Im}\zeta_1(1-\kappa)^{1/2}}{|\zeta_1|}\right]. \tag{4.23}$$

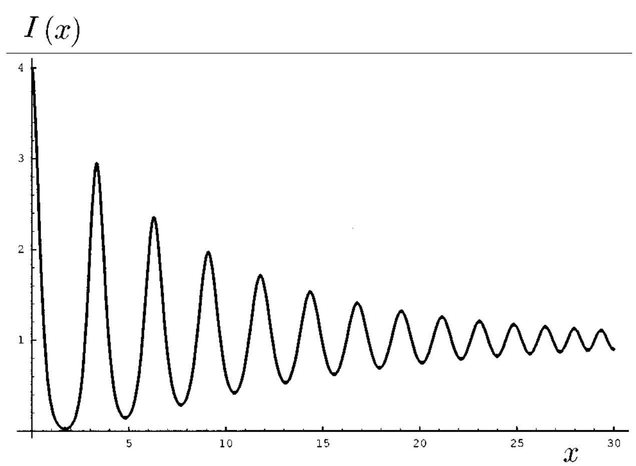

FIG. 1. Transformation of the back edge of an initially steplike seed pulse in a sequence of solitons due to modulation instability without amplification and losses $c=\gamma=0$. Dependence on the intensity $I(x)=|E(x)|^2$ of generated field is found by numerical solution to the Whitham equations for hidden parameter $\zeta_{1,3}$ and shown in arbitrary units.

From Eq. (4.15) it follows that in both limits the phase velocity of waves V_n coincide with V_0, although they have, of course, different values of these limits. Thus, the implicit solution (4.21),(4.22) describes the moving of hidden spectral parameters $\zeta_{2,4}$ from the values associated with the plane-wave solution to those of the asymptotic soliton.

Thus, the change of parameters $\lambda_{1,3}$ describes in the solitonic limit slowly increasing amplitude of generated solitons through relation (4.16). The regime of laser pulse generation consists of two concurrent processes. The first is a formation of solitons near the edge of a steplike pulse and the second is simultaneous amplification of these pulses. Consider the initial conditions associated with soliton formation taking place in attenuators as well. In Fig. 1, transformation of the back edge described by the above solution for hidden parameters is shown. Amplification can cause a significant change in the form of train generated (see Fig. 2), where splitting of steplike pulses with amplification is shown. Note that the regime considered is essentially nonstationary and does not reach stationary asymptotics.

For application to laser systems, the regime of greatest practical utility is that in which the most powerful solitons are generated near the leading edge of the train. Numerics show that the transformation of the steplike pulse into a train of solitons near the leading edge may occur in an attenuator for an initial seed field having special periodic modulation. It was found that such a regime may also occur for some regions of initial values of field amplitude, detuning, and initial nonzero polarization $Q(x,0)$, i.e., for partly inverted systems. These regimes require special experimental efforts and there-

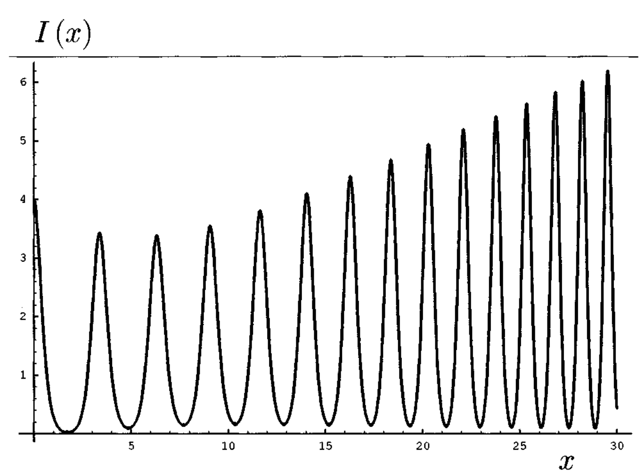

FIG. 2. Transform of the back edge of a steplike pulse for $c'=0.05,\gamma'=0.05$. Dependence on the intensity $I(x)$ is shown in arbitrary units.

fore the region of application of the results is restricted. However, the above-described transformation of the steplike pulse in a train of soliton can be realized for the back edge of pulse as well; see Fig. 2.

The relative contribution of the above mechanism of soliton generation in laser output requires additional investigation. Indeed, the specific property of the Maxwell-Bloch equations is that in a regime of amplification the contribution of the continuous spectrum in an output field may be dominant (cf. [10,7]). The usual experimental setup contains filters removing ''parasite'' spikes appearing after the main pulse; see [4]. In such a system, solitons generated from a steplike seed field can be observed experimentally if the group velocity of the soliton packet differs significantly from that of pulses associated with a continuous spectrum and losses are sufficiently small. To compare relative contributions of the solitonic and nonsolitonic parts in a laser output, let us consider the simplest isolated soliton solution. This one-soliton solution describing the edge of packet is determined by the highest point of the imaginary part of the spectrum and is associated with the initial steplike pulse. Let this point be $\zeta_1=\alpha_0+i\beta_0$; here, α_0,β_0 are real constants. The solution for the amplifying soliton in the above approximation can easily be found by several known methods using the Lax representation for the Maxwell-Bloch model. For instance, the Darboux transform yields, for zero asymptotic $x\to\pm\infty$, a solution for field E as a combination of functions $\Psi_{1,2}=e^{\pm\phi}$, $E=4\beta_0\Psi_1\Psi_2/|\Psi_1|^2+|\Psi_2|^2$, where $\phi=i\int\lambda_1 dx+i\int(dt/4\lambda_1)+b$, b is a constant.

This solution is as follows:

$$E(x,t)=4\beta_0\frac{e^{-i2\,\mathrm{Im}\phi+ia_0}}{\cosh(2\,\mathrm{Re}\phi+a_1)},$$

$$\phi=\frac{\sqrt{2c'x+\zeta_1}(1-e^{-\gamma' t})}{\gamma'}-\frac{\sqrt{2c'x+\zeta_1}-\sqrt{\zeta_1}}{4c'}. \tag{4.24}$$

For solution (4.24) we have $x-c_l t=c_1-\ c_2 t^{-1/2}+o(t^{-1/2})$. Here c_l is the light velocity, $a_{0,2}$ and $c_{1,2}$ are the constants, $c_1=(c_l/\gamma')[1+(\gamma'/4c_l)]$, $c_2=\sqrt{c_l}/\sqrt{8}[1+\gamma'/(4c_l)]$. It is known that a similar solution associated with the continuous spectrum for the above Maxwell-Bloch model (without relaxation) possesses a leading front described approximately by the Bessell functions and depending on a similar variable $\sigma=\sqrt{(c_l t-x)(2x)}$ [11]. For a fixed point σ_0 on the leading edge of a moving pulse, one finds that $x-c_l t=-\sigma_0 t^{-1}+o(t^{-1})$. Here c_1 is a constant. Thus, solitons near the leading edge may be observed for powerful seed pulses and special filters. Solitons and similar solutions will separate after some time due to the difference between corresponding group velocities; therefore, one may observe pure soliton dynamics in lasers with small losses. For nonzero c amplitudes, the solitons generated increase asymptotically as $\sqrt{x}$; see Fig. 2, where amplification is included.

We now describe another mechanism of the laser pulse generation in a quasistationary regime, which may occur if both relaxation and pumping are included. We believe that some experimental results of Ref. [4] can be interpreted within the framework of the mechanism described. We find a stationary regime of laser pulse generation using the above-developed approach. Such a regime can be achieved in gas, ion, and dye laser systems with continuous pumping [4].

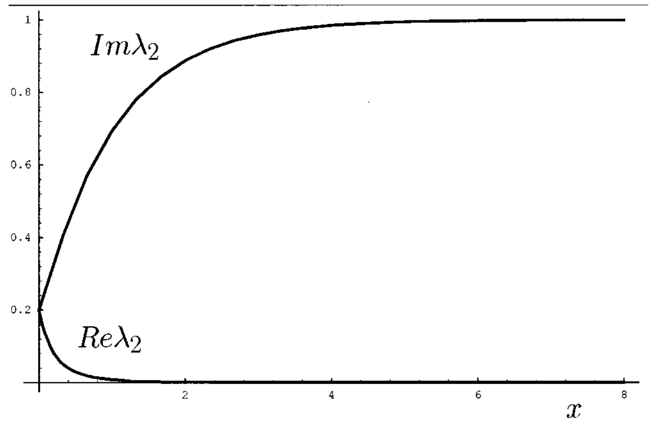

FIG. 3. Dependence of a real and an imaginary part of the spectral parameter $\lambda_2(x)$ on a space coordinate x found numerically. Dependence of λ_4 is the same up to a change of sign of the imaginary part. Imaginary and real parts of $\lambda_2(x)$ tend to their stationary values $\mathrm{Im}\lambda_2(x)\to 1$, $\mathrm{Re}\lambda_2(x)\to 0$, $x\to\infty$.

Assume, as in the above case, that two roots $\lambda_{2,4}$ move from their initial values associated with plane waves to that of the highest leading soliton. The parameters of the highest soliton are determined by the stationary point, i.e., the right-hand side of Eq. (4.14) is equal to zero. Numerous investigations show that both roots $\lambda_{2,4}$ monotonically tends (as the functions of x) to their stationary values λ_s,λ_s^*, respectively, along curves symmetric to the real axis. These stationary values lie in the upper and the lower imaginary part of spectral plane; see Fig. 3. It was found numerically that asymptotics $\lambda_2\to\lambda_s$, $\lambda_4\to\lambda_s^*$; do not depend on the initial conditions. These points λ_s,λ_s^* are the stable foci in a phase plane. If one starts from any small-amplitude plane or periodic wave associated with spectral data $\lambda_{2,4}$, the solution will transform into a form corresponding to values $\lambda_2(\lambda_4)=\lambda_s(\lambda_s^*)$. This transformation is associated with transformation of the modulus of elliptic functions from 0 to 1. This limit corresponds to formation of solitons from the initial plane wave. The latter regime may occur for any initial plane or more complex periodic waves, contrary to the first regime considered above. Such a plane or periodic wave with small amplitude can be used for modeling a noise initiating pulse generation in long lasers.

The transformation of the plane wave into a set of solitons during amplification in the second regime is shown in Fig. 4. Verification of the second mechanism of pulse generation can be achieved experimentally by testing some physical relations. For instance, the experimentally determined relation between the amplitudes of the generated solitons and the values of the pumps could be used as a test. For ion laser pumping, constant C_0 is determined by a current value used for excitation of ions [4]. The relation between the amplitude of the generated pulses and C_0 can be found experimentally. Theoretically, the dependence of a peak intensity of solitons on C_0 can easily be found from the relation determining the stationary point λ_s. We believe that comparison of theoretical results with experimental results found in [4] is satisfactory.

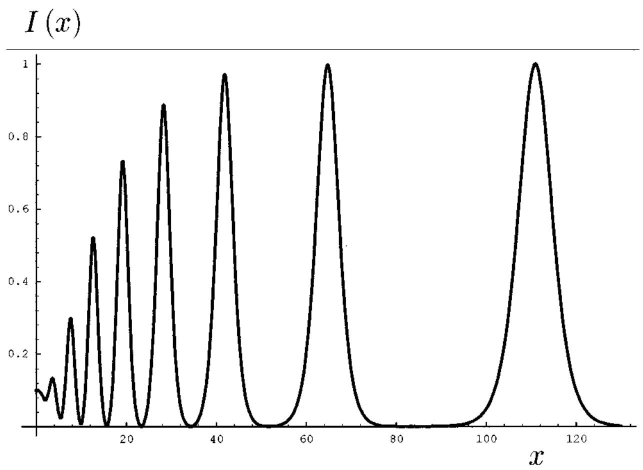

FIG. 4. Transformation of a weak plane wave into a sequence of solitons in the second regime. Dependence on the field intensity $I(x)$ is found numerically and shown in arbitrary units.

In this paper we described two solutions to the Whitham equations and used them for analysis of the generation of soliton trains in some laser schemes. It was found that the second regime more adequately describes known experimental results obtained in studies of the laser systems described here. The above approach can be used to study other near-integrable models having applications in optics. For instance, four-wave-mixing models [17,18] with terms describing relaxation and pumping can be studied without artificial restrictions being imposed on the physical parameters.

The approach developed here can be used in several branches of nonlinear physics. For instance, one can study small deformations of chiral fields without restrictions being imposed by an exact integrability condition. Lax representation for the latter model is as follows:

$$D_\xi \Phi = \frac{u}{\lambda - 1}\Phi, \quad D_\eta \Phi = \frac{v}{\lambda + 1}\Phi. \tag{4.25}$$

Here u,v are the matrix functions and λ is a spectral parameter. This deformation can be used in some gravitation theory [19]. A corresponding theory was developed by Belinsky and Zakharov [19] for soliton solutions. But direct application of IST with variables depending spectral parameter have not, to our knowledge, been performed for periodic waves.

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